UDC 532.5

FLOW PAST A PROFILE IN A MAGNETIC FIELD PERPENDICULAR TO THE STREAM PLANE

PMM Vol. 36, №2, 1972, pp. 232-238

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(Received August 4, 1971)

The problem of flow of an inviscid conducting fluid past an arbitrary profile in a magnetic field perpendicular to the stream plane is reduced to a system of integral equations. An exact solution is derived for the case of an irrotational flow, and is illustrated on the example of flow past a circular cylinder. The asymptotic solution of the problem is examined at high magnetic Reynolds numbers.

For the problem considered here the equations of hydrodynamics for a laminar potential flow can be written in the form

$$\Delta H - \nu_m^{-1}(\mathbf{v}\nabla H) = 0 \qquad (\nu_m = 1 / \mu\mu_0\sigma) \tag{1}$$

$$\frac{v^2}{2} + \frac{P}{\rho} + \frac{\mu\mu_0 H^2}{2\rho} = \text{const}, \qquad \operatorname{rot} \mathbf{v} = 0, \qquad \operatorname{div} \mathbf{v} = 0$$

where ν_m is the magnetic viscosity, μ and μ_0 are the magnetic permeabilities of fluid and vacuum, respectively, and σ is the conductivity of fluid.

The boundary conditions for velocity $\mathbf{v}$ are the same as in the absence of a magnetic field. Because of this it follows from (1) that the motion of fluid is not affected by a magnetic field which remains perpendicular to the plane of flow, and the problem reduces to the determination of the distortion of that field by a given stream, i.e., to the determination of H appearing in the first of Eqs. (1) for a known function $\mathbf{v}$ (x, y).

To analyze the solution of this problem independently of the profile form we pass from variables x, y to Φ (x, y) and Ψ (x, y) ($W = \Phi + i\Psi$ is the known complex potential of the stream). The first of Eqs. (1) expressed in terms of variables Φ and Ψ is of the form

$$\frac{\partial^2 H}{\partial \Phi^2} + \frac{\partial^2 H}{\partial \Psi^2} - \frac{1}{\nu_m}\frac{\partial H}{\partial \Phi} = 0 \tag{2}$$

The flow region D outside the contour C mapped onto the plane Φ, Ψ (region D^*) sectioned along the real semiaxis $\Phi > 0$, is shown in Fig. 1 together with corresponding points [of the profile]. Circulation $\Gamma = \Phi_{B-*} - \Phi_{B+*}$ around the aerofoil is assumed to be specified. Conformal mapping of D onto D^* is carried out by means of function W $(z) = \Phi + i\Psi$.

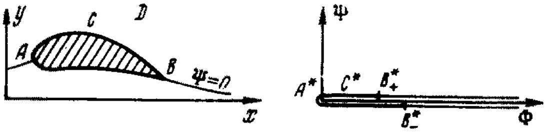

Fig. 1.

The method of variables Φ, Ψ had been used some time ago by Boussinesq [1] in the problem of heat convection from a cylinder, and later applied in [2, 3] to the analysis of magnetohydrodynamic flows over semi-infinite bodies. The problem of heat convection in a potential stream [4], that of the flow of a viscous conducting fluid along the generatrix of a cylinder in a transverse magnetic field [5], and others reduce to equations of the kind of (2). However for $\Gamma \neq 0$ the specific properties of region D^* in the plane Φ, Ψ inhibit the application of solutions derived in [1 - 5] to this problem in the case of arbitrary numbers R_m.

Let us formulate the magnetic field boundary conditions. First, it should be noted that the first of Eqs. (1) defines the magnetic field H correct to within a certain constant H_0 which can be considered to be the external field specified at infinity. Hence it is sufficient to specify the boundedness of H at some distance from the profile as the condition at infinity.

We assume that a thin conducting layer [of fluid] of variable thickness δ whose conductivity is σ^e, lies along the profile surface and that an external source of current of specified potential U at terminals is switched to this layer. Under the latter there is an insulator (the "body" of the profile) whose surface is defined by equation $n = -\delta(s)$, where n is the coordinate along the outward normal to the contour C, and s is the arc coordinate (Fig. 2).

The assumption that the conducting layer is very thin implies that everywhere $\delta(s) \ll L$, where L is a characteristic dimension along s (e.g., the radius of curvature of profile C). Since throughout the fluid the components of current density $\mathbf{j}$ are of the form $j_s = \partial H / \partial n$ and $j_n = -\partial H / \partial s$, from the continuity of tangential components of the electrical field $\mathbf{E} = \mathbf{j}/\sigma - \mu\mu_0 [\mathbf{v} \times \mathbf{H}]$ we obtain

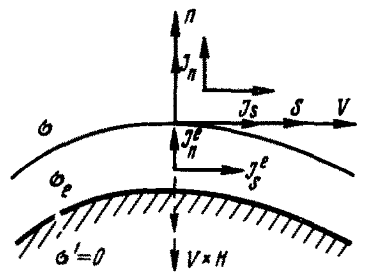

Fig. 2.

$$\left(\frac{\partial H}{\partial n}\right)_c = \frac{\sigma}{\sigma^e}[j_s^e]_{n=0} \qquad (3)$$

where j_s^e is the tangential component of current density in the conducting layer region $(-\delta(s) \leqslant n \leqslant 0)$. In the conducting layer region equations $\operatorname{div} \mathbf{j}^e = 0$ and $\operatorname{rot} \mathbf{E}^e = 0$ are, owing to $\delta(s) \ll L$, valid throughout that layer and, furthermore, the positions of current sources can be defined by

$$\frac{\partial}{\partial n}(\sigma^e E_n^e) + \frac{\partial}{\partial s}(\sigma^e E_s^e) = 0, \qquad \frac{\partial E_n^e}{\partial s} - \frac{\partial E_s^e}{\partial n} = 0 \qquad (4)$$

Since the layer is thin, from the second of Eqs. (4) we have $\partial E_s^e / \partial n = 0$, hence

$$E_s^e = E_s^e(s) \qquad (5)$$

Integrating the first of relationships (4) over the layer $(-\delta(s) \leqslant n \leqslant 0)$ and taking into consideration (5) and the boundary condition for j_n^e

$$(j_n^e)_{n=0} = (j_n)_c = -\left(\frac{\partial H}{\partial s}\right)_c, \qquad (j_n^e)_{n=-\delta(s)} = 0$$

we obtain

$$\left(\frac{\partial H}{\partial s}\right)_c = \delta(s)\frac{d}{ds}(\sigma^e E_s^e)$$

Taking into account (3), and since E_s^e (consequently also j_s^e) do not vary across the conducting layer, we have

$$\left(\frac{\partial H}{\partial s}\right)_c = \delta(s)\frac{d}{ds}\left[\frac{\sigma^e}{\sigma}\left(\frac{\partial H}{\partial n}\right)_c\right] \tag{6}$$

Hence, assuming that $H\ (x,\ y)$ is continuous up to the contour C and that in the following $\delta = \text{const}$, we obtain

$$\left(H - \frac{\delta\sigma^e}{\sigma}\frac{\partial H}{\partial n}\right)_c = C_0 \tag{7}$$

This relationship represents the boundary condition for our problem. Constant C_0 can be defined in terms of potential U at the source terminals by

$$U = \oint\limits_c E_s^e\, ds = \oint\limits_c \frac{1}{\sigma}\left(\frac{\partial H}{\partial n}\right)_c ds \tag{8}$$

Relationships similar to (6) were considered in [6, 7].

Let us examine two limit cases of the general boundary condition (7). Let l be the characteristic thickness of the fluid layer surrounding the profile in which H and ∂H / / ∂n are of the same order. It follows from the induction equation that

$$l = L / \sqrt{R_m}, \qquad R_m = v_\infty L / \nu_m = v_\infty L \mu_0 \mu \sigma$$

where R_m is the magnetic Reynolds number. It follows from (7) that when

$$\frac{\delta\sigma^e}{l\sigma} = \frac{\sigma^e \delta \sqrt{R_m}}{L\sigma} \ll 1$$

the boundary condition must be $(H)_c = C_0$. But then, according to the principle of maximum [8], $H = C_0$ everywhere outside contour C. This trivial solution exists, if the external magnetic field is equal C_0

We note that in this limit case $(j_n)_c = 0$, hence there is no "penetration" of the current induced in the fluid into the conducting layer, as if the profile were insulated from that current. Moreover such currents cannot even arise, since the specified current circulating in the closed conducting layer cannot generate a field external to contour C

The opposite is true in the other limit case in which

$$\delta\sigma^e \sqrt{R_m} / L\sigma \gg 1$$

and the condition

$$(\partial H / \partial n)_C = U\sigma / L_c$$

where L_c is the length of the conducting layer (contour C), is taken as the boundary condition, as implied by (7) and (8). In this case the most intensive exchange of current between the conducting layer and the fluid stream takes place.

In the following we shall deal specifically with this case of the boundary condition. On part C^* of the boundary of region D^* corresponding to the streamlined profile we have

$$\left.\frac{\partial H}{\partial \Psi}\right|_{C*} = \frac{U\sigma}{L_c}\left.\sqrt{\left(\frac{\partial x}{\partial \Phi}\right)^2 + \left(\frac{\partial x}{\partial \Psi}\right)^2}\right|_{\Psi=0} = G(\Phi, 0) \tag{9}$$

For solving the problem in the Φ, Ψ-plane it is necessary to specify in addition to the boundary condition (9) the conditions at the remaining part of the boundary of region D^*, i.e., at the extremities of the cross section to which in the x, y-plane corresponds the streamline $\Psi\ (x,\ y) = 0$ extending from the rear stagnation point of the profile to infinity. It is reasonable to assume that along this line the magnetic field H and its first derivatives are continuous. On this assumption we obtain the following conditions:

$$H\Big|_{\substack{\Psi=+0\\ \Phi=\Phi}} = H\Big|_{\substack{\Psi=-0\\ \Phi=\Phi+\Gamma}}; \qquad \frac{\partial H}{\partial \Psi}\Big|_{\substack{\Psi=+0\\ \Phi=\Phi}} = \frac{\partial H}{\partial \Psi}\Big|_{\substack{\Psi=-0\\ \Phi=\Phi+\Gamma}} \qquad \text{for } \Phi > \Phi_{B+*}$$

We also assume that field H is bounded at infinity.

We introduce dimensionless variables $\varphi = \Phi \, / \, (v_\infty L)$, $\psi = \Psi \, / \, (v_\infty L)$ and $h = H \, / \, H_0$ pass from $h\ (\varphi, \psi)$ to the new function

$$g(\varphi, \psi) = h(\varphi, \psi)\, e^{-R\varphi}$$

and in polar coordinates $\rho = \sqrt{\varphi^2 + \psi^2}$, $\theta = \operatorname{arc\,tg} \psi/\varphi$ obtain

$$\Delta g - R^2 g = 0 \tag{10}$$

$$\frac{\partial g}{\partial \theta}\Big|_{\theta=0} = \rho G(\rho, 0)\, e^{-R\rho} \qquad (0 \leqslant \rho \leqslant a)$$

$$\frac{\partial g}{\partial \theta}\Big|_{\theta=2\pi} = -\rho G(\rho, 0)\, e^{-R\rho} \qquad (0 \leqslant \rho \leqslant a+\gamma)$$

$$g(\rho, 0) = e^{R\gamma} g(\rho+\gamma, 2\pi) \qquad (\rho > a) \tag{11}$$

$$\frac{1}{\rho}\frac{\partial g}{\partial \theta}\Big|_{\theta=0} = \frac{e^{R\gamma}}{\rho+\gamma}\frac{\partial g}{\partial \theta}\Big|_{\theta=2\pi} \qquad (\rho > a)$$

where $R = R_m \, / \, 2 = v_\infty L \, / \, (2 \nu_m)$, $\gamma = \Gamma \, / \, (v_\infty L)$ and $a = \varphi\,(x_{B+}, y_{B+})$.

The solution of Eq. (10) bounded at infinity can be written as

$$g(\rho, \theta) = \int_0^\infty \left[A(\lambda)\frac{\operatorname{sh}(2\pi - \theta)\lambda}{\operatorname{sh} 2\pi\lambda} + B(\lambda)\frac{\operatorname{sh}\lambda\theta}{\operatorname{sh} 2\pi\lambda}\right] K_{i\lambda}(R\rho)\, d\lambda \tag{12}$$

where $K_{i\lambda}\ (x)$ is the Macdonald function of the imaginary index.

The stipulation that solution $g\ (\rho, \ \theta)$ must satisfy the four conditions (11) yields

$$\int_0^\infty \frac{\lambda}{\operatorname{sh} 2\pi\lambda}(B - A\operatorname{ch} 2\pi\lambda)\, K_{i\lambda}(R\rho)\, d\lambda = \rho G(\rho, 0)\, e^{-R\rho} \qquad (0 \leqslant \rho \leqslant a)$$

$$\int_0^\infty \frac{\lambda}{\operatorname{sh} 2\pi\lambda}(B\operatorname{ch} 2\pi\lambda - A)\, K_{i\lambda}(R\rho)\, d\lambda = -\rho G(\rho, 0)\, e^{-R\rho} \qquad (0 \leqslant \rho \leqslant a+\gamma) \tag{13}$$

$$\int_0^\infty A K_{i\lambda}(R\rho)\, d\lambda = e^{R\gamma}\int_0^\infty B K_{i\lambda}[R(\rho+\gamma)]\, d\lambda \qquad (\rho > a)$$

$$\frac{1}{\rho}\int_0^\infty \frac{\lambda}{\operatorname{sh} 2\pi\lambda}(B - A\operatorname{ch} 2\pi\lambda)\, K_{i\lambda}(R\rho)\, d\lambda =$$

$$= \frac{e^{R\gamma}}{\rho+\gamma}\int_0^\infty \frac{\lambda}{\operatorname{sh} 2\pi\lambda}(B\operatorname{ch} 2\pi\lambda - A)\, K_{i\lambda}[R(\rho+\gamma)]\, d\lambda \qquad (\rho > a)$$

These relationships constitute a system of integral equations for the determination of functions $A\ (\lambda)$ and $B\ (\lambda)$. It seems that for a specific stream this system can generally be solved only by numerical methods. The determination of functions $A\ (\lambda)$ and $B\ (\lambda)$ is the simplest in the case of irrotational flow past a profile.

In fact, by setting in (13) $\gamma = 0$, after transformation we obtain

$$\int_0^\infty (A-B) K_{i\lambda}(R\rho)\, d\lambda = 0 \qquad (\rho > a_0, \quad a_0 = a|_{\gamma=0})$$

$$\int_0^\infty (A+B) \lambda \operatorname{th} \pi\lambda K_{i\lambda}(R\rho)\, d\lambda = -2\rho G(\rho, 0) e^{-R\rho} \qquad (0 \leqslant \rho \leqslant a_0) \qquad (14)$$

$$\int_0^\infty (A+B) \lambda \operatorname{th} \pi\lambda K_{i\lambda}(R\rho)\, d\lambda = 0 \qquad (\rho > a_0)$$

Owing to the symmetry of the problem formulation in the φ, ψ-plane, the magnetic field is the same at points of contour C^* situated at both extremities of the cross section, i.e.,

$$g(\rho, 0) = g(\rho, 2\pi) \qquad (\rho \in C^*)$$

Hence

$$\int_0^\infty (A-B) K_{i\lambda}(R\rho)\, d\lambda = 0 \qquad (0 \leqslant \rho \leqslant a_0) \qquad (15)$$

From (15) and the first of Eqs. (14) follows that $A(\lambda) = B(\lambda)$. The last two of Eqs. (14) can now be used for the determination of function $A(\lambda)$. Using the Kantorovich-Lebedev transformation [9], we obtain

$$A(\lambda) = -\frac{2}{\pi^2} \operatorname{ch} \pi\lambda \int_0^{a_0} G(t) e^{-Rt} K_{i\lambda}(Rt)\, dt \qquad (16)$$

Substituting this expression into (12) and applying the known convolution formula

$$\int_0^\infty \operatorname{ch}[(\pi - \theta)\lambda] K_{i\lambda}(x) K_{i\lambda}(y)\, d\lambda = \frac{\pi}{2} K_0 (\sqrt{x^2 + y^2 - 2xy\cos\theta})$$

we obtain

$$g(\rho, \theta) = -\frac{1}{\pi} \int_0^{a_0} G(t) e^{-Rt} K_0 (R\sqrt{t^2 + \rho^2 - 2t\rho\cos\theta})\, dt$$

For a dimensionless magnetic field we have

$$h(\varphi, \psi) = -\frac{1}{\pi} \int_0^{a_0} G(t) e^{R(\varphi - t)} K_0 (R\sqrt{\psi^2 + (\varphi - t)^2})\, dt \qquad (17)$$

This solution coincides to within notation with that of the problem of heat convection from a cylinder in a potential stream derived in [4] by the method of sources and investigated in the case of small R_m $(=2R)$ numbers.

As an example of the use of solution (17) let us consider the irrotational flow past a cylinder for arbitrary R_m. In this case

$$G(\varphi, 0) = \frac{U\sigma}{2L_c V_\infty \sqrt{2\varphi - \varphi^2}}, \qquad a_0 = 2$$

and from (17) we have

$$h(\varphi, \psi) = -\frac{U\sigma e^{R\varphi}}{L_c \pi V_\infty} \int_0^{\pi/2} e^{-2R\cos^2 u} K_0 (R\sqrt{\psi^2 + (\varphi - 2\cos^2 u)^2})\, du$$

Using Simpson's quadratic rule we calculate $h(\varphi, \psi)$ and then, applying formulas

$$\varphi = (r + 1/r)\cos\chi + 2, \qquad \psi = (r - 1/r)\sin\chi$$
$$r^2 = x^2 + y^2, \qquad \chi = \operatorname{arctg} y/x$$

construct for the cylinder the lines of level $h(x, y)$ for $R_m = 1$ and $R_m = 2$ shown in Fig. 3 by solid and dotted lines, respectively. These show that the gradient of the magnetic field in the layer adjacent to the cylinder surface increases with increasing R_m. This implies the formation of a magnetic boundary layer for $R_m \gg 1$. The examination of the asymptotic behavior of the solution of Eq. (10) for $R_m \gg 1$ is, therefore, of some interest. Introducing the small parameter $\varepsilon = 1 / R \ll 1$ and passing in Eq. (10) to variables ξ and η related to φ and ψ by formulas $\varphi = \xi^2 - \eta^2$ and $\psi = 2\xi\eta$, we obtain

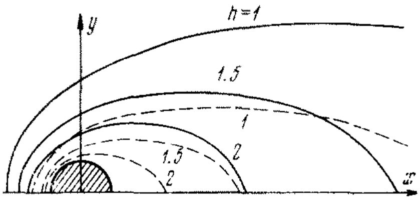

Fig. 3

$$\varepsilon^2 \left(\frac{\partial^2 g}{\partial \xi^2} + \frac{\partial^2 g}{\partial \eta^2} \right) - 4(\xi^2 + \eta^2) g = 0 \tag{18}$$

Region D^* shifts to the upper half-plane $\eta > 0$ and contour C^* becomes segment $(-\sqrt{a+\gamma}, \sqrt{a})$ of the ξ-axis. The condition

$$\left. \frac{\partial g}{\partial \eta} \right|_{\eta=0} = 2\xi G(\varphi(\xi), 0)\, e^{-\xi^2/\varepsilon} \tag{19}$$

which is implied by (11) must be satisfied along this segment. To find a boundary value kind of solution of Eq. (18) with condition (19) we set, in accordance with the generally accepted method [10], $\xi = \xi$ and $\eta = \varepsilon t$. Neglecting terms of the order of ε^2 from (18) and (19) we obtain

$$g^\circ(\xi, t) = -\frac{\varepsilon\xi}{|\xi|} G(\varphi(\xi), 0) \exp\left(-\frac{\xi^2 + 2|\xi| t\varepsilon}{\varepsilon} \right)$$

Reverting to $h(\varphi, \psi)$, we derive the asymptotic value $h^\circ(\varphi, \psi)$ for field h at $R_m \gg 1$

$$h(\varphi, \psi) \sim h^\circ(\varphi, \psi) = \pm \varepsilon(\varphi, 0)\, e^{-|\psi|/\varepsilon} \tag{20}$$

It will be seen from (20) that the thickness of the magnetic boundary layer is of the order of $1 / \sqrt{R_m}$.

Authors express their sincere thanks to N. I. Akhiezer for his interest in this work, and to G. A. Liubimov and S. A. Regirer for their valuable comments.

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Translated by J. J. D.

UDC 533.697.4

THREE-DIMENSIONAL SUB- AND SUPERSONIC FLOWS IN NOZZLES AND CHANNELS OF VARYING CROSS SECTION

PMM Vol. 36, №2, 1972, pp. 239-247
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(Received July 22, 1971)

Three-dimensional sub- and supersonic flows of gas in nozzles and channels of varying cross section are analyzed. The inverse problem of the theory of Laval nozzles is formulated and extended to three-dimensional flows. An implicit three-point difference scheme with varying pitch along a layer is proposed. In the neighborhood of the surface for which the Cauchy data are specified an asymptotic series expansion in terms of the stream-function is derived and the method of solving related equations is indicated. Examples of calculations of three-dimensional flows in nozzles are presented. Papers [1 - 3] dealing with three-dimensional supersonic flows in nozzles and paper [4] in which an analytical solution is derived for the flow in the neighborhood of the nozzle center should be noted among recent publications.

1. Fundamental equations and statement of problem. We introduce a system of curvilinear coordinates linked with the curve $y = f_0(s)$ lying in the xy-plane. The coordinates of a point are defined in this system by the arc length s, the distance r along the normal to this curve, and by the angle φ in a plane normal to it (Fig. 1).

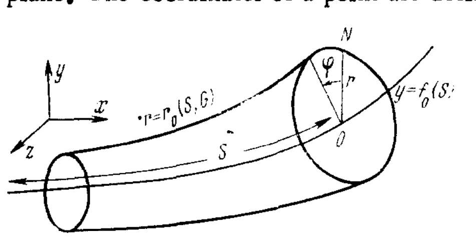

Fig. 1

We transform the equations of gasdynamics in the system of coordinates s, r and φ [5] by passing to new independent variables ψ and θ such that ψ = const and θ = const represent stream surfaces which can be introduced for analyzing three-dimensional